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FUNCTION ALGEBRAS

BY

H. L. ROYDEN

TECHNICAL REPORT NO. 21

January 21, 1963

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U. S. ARMY RESEARCH OFFICE

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APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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FUNCTION ALGEBRAS

by

H. L. ROYDEN

<u>Introduction</u>: By a function algebra I shall mean a collection A of complex-valued functions on a set X such that the (pointwise) sum and the product of two functions in A are again in A. We shall always suppose that A contains the constant functions so that A becomes an algebra with unit over the field of complex numbers. A function algebra A is called self adjoint if the complex conjugate of each function in A is again in A, and the theory of self-adjoint algebras is quite different from that of non-self adjoint algebras. An example of a self-adjoint function algebra is given by the algebra of all continuous complex-valued functions on a topological space X. A thorough description of the theory of such algebras is given in the book by Gilman and Jerison [27].

My own interest in function algebras arose from the study of algebras of analytic functions on some sort of an analytic space X. These algebras are of course very far from being self-adjoint. A considerable amount of effort has gone into the study of certain of these algebras and of the relationship between algebraic properties of such an algebra and the analytic structure of the underlying space X. This

includes work by Bers [8], Bishop [10], [12], Chevalley and Kakutani [43], Edwards [22], Heins [29], Helmer [30], Henriksen [33], Rudin [57], Wermer [70], [71] and myself [54], [55]. One of the principal purposes of the present discussion is to give a general treatment of function algebras into which many of the above results may be fitted.

Another purpose is to illustrate the application of methods from the theory of functions of several complex variables to derive results about non-self adjoint function algebras. Excellent examples of such applications are given in the papers by Arens and Calderon [2] and Rossi [52], and in Section 7 I use a theorem of Arens and mine to illustrate these techniques. This theorem describes the one-dimensional cohomology of the spectrum of a Banach algebra in terms of the group of invertible elements of the algebra. It generalizes to arbitrary commutative Banach algebras the theorems of Bruschlinsky [18] and Eilenberg [23], describing the one-dimensional cohomology of a compact set X in terms of $C(X)$.

The topics discussed here are influenced by my own interest, and because of the excellent summary by Wermer [76] I have omitted discussion of much of the work of Wermer, Bishop, and Rossi.

1. Algebras and their spectra. Let X be a set of points and A an algebra of complex valued functions on X. We shall always assume that 1 belongs to A and that the functions of A separate the points of X. If X is a topological space, we shall also suppose that each element in X is continuous.

By the spectrum $\Sigma = \Sigma(A)$ of A we mean the set of all non-zero homomorphisms of A into the complex field. The set X is imbedded in Σ if we identify each $x \in X$ with the homomorphism $f \to f(x)$. There is a natural isomorphism of A with an algebra $\hat{A}$ of functions on Σ defined by $\hat{f}(\pi) = \pi(f)$ for $\pi \in \Sigma$ and $f \in A$. The algebra A then consists of the restrictions to X of the elements of $\hat{A}$.

It is often convenient to topologize Σ by using the weakest topology under which the elements of $\hat{A}$ are continuous. We refer to this topology as the Gelfand topology for Σ. It is always Hausdorff, and the natural embedding of X into Σ with the Gelfand topology is a continuous mapping, although the topology of X may be stronger than the topology it would have as a subspace of Σ with the Gelfand topology. If X is compact, however, this natural embedding must be a homeomorphism into Σ with the Gelfand topology. The following lemma gives a condition for the compactness of Σ in the Gelfand topology:

Lemma 1. The spectrum Σ of a function algebra A is compact in the Gelfand topology if and only if each element of $\hat{A}$ is bounded on Σ.

Another entity which can be associated with A is the set $\mathfrak{M}$ of maximal ideals of A. Since each element of Σ is uniquely characterized by its kernel, which is a maximal ideal, we may consider Σ to be (identified with) a subset of $\mathfrak{M}$. Proposition 3 gives a condition for this subset to be all of $\mathfrak{M}$.

2. Divisibility properties of function algebras. In this section we explore some of the consequences of various relations between zeros of elements of a function algebra and their divisibility properties. Here we consider the spectrum Σ of an algebra as a set without a topology. Some of the properties which a function algebra A on a set X may possess are the following:

(α_o) If $f \in A$, and $\sup_{x \in X} |f(x)| < 1$, then $1 - f$ has an inverse in A.

(α) If $f \in A$ and if f is never zero, then f has an inverse in A.

(β) If $f_1, \ldots, f_n$ are elements of A with no common zeros, then there are elements $g_1, \ldots, g_n$ in A such that $g_1 f_1 + \cdots + g_n f_n = 1$.

(γ) If $f \in A$ and f is not identically zero, there are a finite number of functions $f_1, \ldots, f_n$ in A which separate the zeros of f.

If A has property (α) we say that it is inverse closed. If it has the property (α_o) we say that it is weakly inverse closed. Note that (β) implies (α) and (α) implies (α_o). If A is closed under uniform convergence, then (α_o) holds.

Proposition 1. If the algebra A has property (α_o), then for each $f \in A$ and each $\pi \in \Sigma(A)$ we have

$$|\pi f| \leq \sup_{x \in X} |f(x)| .$$

Proof: Suppose that for all $x \in X$ we have $|f(x)| \leq M < \infty$. Then for each complex number λ with $|\lambda| > M$, we have $\sup |\lambda^{-1} f| < 1$, and so $1 - \lambda^{-1} f$ has an inverse g in A. Thus $1 = \pi(1) = \pi[g(1 - \lambda^{-1} f)] = \pi(g)\ \pi(1 - \lambda^{-1} f) = \pi(g)\ [1 - \lambda^{-1}\ \pi(f)]$, and consequently $0 \neq 1 - \lambda^{-1}\ \pi(f)$. But this means $\pi f \neq \lambda$. Since this holds for every λ with $|\lambda| > M$, we must have $|\pi f| \leq M$, proving the proposition.

Combining this proposition with Lemma 1 we have the following corollary:

Corollary: Let A be a function algebra with property (α_o). Then the spectrum $\Sigma(A)$ is compact (in the Gelfand topology) if and only if each f in A is bounded.

A slight modification of the proof of Proposition 1 gives us the following consequence of property (α):

Proposition 2: Let A have property (α), and let $\pi \in \Sigma(A)$. Then

$$\pi f \in f[X] .$$

The following proposition gives some evidence of the effect of completion of a function algebra on its spectrum.

Proposition 3: Let A be an algebra of bounded functions having property (α_o). Then the spectrum and maximal ideal space of A coincide with the spectrum of $\overline{A}$, the completion of A under uniform convergence on X. Moreover, the Gelfand topologies of $\Sigma(A)$ and $\Sigma(\overline{A})$ also coincide.

Proof: Proposition 1 implies that each homomorphism π in $\Sigma(A)$ is continuous in the sup norm for A (i.e., the norm defined by $\|f\| = \sup_{x \in X} |f(x)|$). Hence each $\pi \in \Sigma(A)$ has a unique extension to a homomorphism in $\Sigma(\overline{A})$. Conversely, each π in $\Sigma(\overline{A})$ is the unique continuous extension of π restricted to A. Thus $\Sigma(A)$ and $\Sigma(\overline{A})$ coincide. The Gelfand topology for $\Sigma(\overline{A})$ is by its definition stronger than the Gelfand topology for $\Sigma(A)$. Since the latter is Hausdorff, while the former is compact by the corollary of Proposition 1, the two topologies are the same.

Let I be a maximal ideal of A. It follows from (α_o) that the completion $\overline{I}$ in $\overline{A}$ of I is an ideal which does not contain 1. From the fact that $\overline{A}$ is a Banach algebra we deduce [25] that there is a homomorphism of $\overline{A}$ onto C whose kernel contains $\overline{I}$. The restriction of this homomorphism to A has kernel I, and so the set of maximal ideals of A coincides with the spectrum of A, proving the proposition.

Proposition 4: Let A be an algebra of functions on a set X. If X is the set $\mathcal{M}$ of maximal ideals of A, then (β) holds.

Proof: Let $f_1, \dots, f_n$ be a finite set of functions such that 1 cannot be expressed as $\Sigma\, e_i f_i$. Then the ideal generated by $f_1, \dots, f_n$ is proper and thus contained in a maximal ideal. By hypothesis there is a point x such that this maximal ideal consists of all functions vanishing at x. Thus x is a common zero of $f_1, \dots, f_n$, and consequently (β) holds proving the proposition.

Proposition 5: Let A be an algebra of continuous functions on a compact space X. Then X is the spectrum of A if and only if A has property (β).

Proof: If X is the spectrum of A, then X is also the maximal ideal space of A by Proposition 3. Hence (β) holds by Proposition 4.

Let us assume on the other hand that A has property (β), and let I be a maximal ideal of A. Then (β) implies that any finite set of functions of I have a common zero. Since X is compact, this implies all functions of I have a common zero x. Since I is maximal, it must be the ideal consisting of all functions vanishing at x. Thus $\Sigma(A)$ coincides with X, proving the proposition.

Proposition 6: Let A be a function algebra on a set X, and suppose that A has properties (β) and (γ). Then X is the spectrum of A.

Proof: Let π be an element of the spectrum of A and f_o an element of the kernel of π which is not identically zero. (Such an element must exist unless X consists of a single point.) By (γ) there are functions $f_1, \ldots, f_n$ which separate points of the set $\{x : f_o(x) = 0\}$. Subtracting suitable constants, we may take the functions $f_1, \ldots, f_n$ to be in the kernel of π. Since (β) implies that any finite set of functions in a proper ideal have a common zero, there is a point $x \in X$ such that $f_i(x) = 0$, $0 \le i \le n$. On the other hand the functions $f_o, f_1, \ldots, f_n$ cannot vanish simultaneously at any other point of X because of the fact that $f_1, \ldots, f_n$ separate the zeros of f_o. If now g is any function in the kernel of π, (β)

implies that g must have a zero in common with $f_o, f_1, \dots, f_n$. Hence $g(x) = 0$. But for any $g \in A$, the function $g - \pi g$ is in the kernel of π, whence $g(x) = \pi g$. Thus π is the evaluation at x, proving the proposition.

3. Homomorphisms of function algebras. Let A and B be two function algebras on topological spaces X and Y, and let h be a homomorphism of A into B. Then for each homomorphism $\pi \in \Sigma(B)$ we have a homomorphism $\varphi(\pi)$ in $\Sigma(A)$ defined by $\varphi(\pi) = \pi \circ h$. Thus for each $y \in Y$ and $f \in A$ we have $f(\varphi(y)) = (hf)(y)$. We call the mapping φ the adjoint of h (and sometimes write $\varphi = h^*$). If we use the Gelfand topology for $\Sigma(A)$, then φ is a continuous map of $\Sigma(B)$ into $\Sigma(A)$, and its restriction to Y is a continuous map of Y into $\Sigma(A)$.

If we know that the space X, considered as a point set without topology is the spectrum of A, then the adjoint mapping φ of h is defined as a mapping of Y into X. Under what conditions can we say that it is continuous? If X is compact, then it must already have the Gelfand topology, and so φ is continuous. We state this as the following proposition:

Proposition 7: Let A be a function algebra on a compact set X, and suppose that X, as a point set, is the spectrum of A. Let B be a function algebra on a topological space Y, and let h be a homomorphism of A into B. Then there is a unique continuous map φ of Y into X such that

$$(hf)(y) = f(\varphi(y)) .$$

Even when X is not compact and does not have the Gelfand topology we can sometimes insure the continuity of φ as follows: For a function algebra A on X we define the following condition:

(δ) The space X is metrizable, and for each sequence $< x_n >$ from X which does not converge there is an f in A such that the sequence $< f(x_n) >$ does not converge.

Proposition 8: Let A and B be function algebras on the metrizable spaces X and Y. Suppose that X (as a point set) is the spectrum of A and that condition (δ) holds. Then there is a unique continuous mapping φ of Y into X such that $(hf)(y) = f(\varphi(y))$.

Proof: Only the continuity of φ needs to be established. Since X and Y are metrizable, it suffices to show that φ takes convergent sequences into convergent sequences. Let $< y_n >$ be a convergent sequence in Y with limit y. Then for each $f \in A$ we have $f(\varphi(y_n)) = (hf)(y_n)$, which converges to $(hf)(y) = f(\varphi(y))$. Since this holds for each $f \in A$, condition (δ) implies that the sequence $< \varphi(y_n) >$ converges to some element x in X. Since $f(x) = f(\varphi(y))$ for all $f \in A$ and A separates points of X, we have $x = \varphi(y)$. Thus φ is continuous, proving the proposition.

An extension of the mapping properties in these propositions can be gotten as follows: Let D be a directed set under a quasi-order $\prec$, and suppose that to each $\alpha \in D$ there is assigned an algebra A_α and that to each pair of elements $\alpha \prec \beta$ in D there is a homomorphism $h_\alpha^\beta : A_\alpha \to A_\beta$ such that for each α we have h_α^α the identity and for $\alpha \prec \beta \prec \gamma$ we have $h_\beta^\gamma \circ h_\alpha^\beta = h_\alpha^\gamma$. Then the system $\{A_\alpha, h_\alpha^\beta\}$ is called a <u>direct system</u> of algebras. For each such direct system there is an algebra A called the direct limit of the system and a family of homomorphisms $h_\alpha : A_\alpha \to A$ such that

(i) $$h_\beta h_\alpha^\beta = h_\alpha \qquad \text{for } \alpha \prec \beta$$

(ii) $$A = \bigcup_\alpha \text{Range } h_\alpha .$$

These two properties characterize A to within isomorphism.

If we suppose that to each α we have assigned a topological space X_α and that to each pair $\alpha \prec \beta$ we have assigned a continuous mapping $\varphi_\alpha^\beta : X_\beta \to X_\alpha$ such that φ_α^α is the identity and $\varphi_\alpha^\beta \varphi_\beta^\gamma = \varphi_\alpha^\gamma$ for $\alpha \prec \beta \prec \gamma$, then we call $\{X_\alpha, \varphi_\alpha^\beta\}$ an <u>inverse</u> system of topological spaces. For each such inverse system we define a topological space X, called the inverse limit of the system, by taking X to consist of those elements $x = \{x_\alpha\}$ of the direct product $\bigtimes_D X_\alpha$ which satisfy $\varphi_\alpha^\beta(x_\beta) = x_\alpha$ for $\alpha \prec \beta$. We define the projection $\varphi_\beta : X \to X_\beta$ by setting $\varphi_\beta(x) = x_\beta$. We have $\varphi_\alpha = \varphi_\alpha^\beta \varphi_\beta$ for $\alpha \prec \beta$. The topology in X may be defined either by considering X as a subspace of the topological product $\bigtimes X_\alpha$ or equivalently by taking the weakest topology such that the projections φ_α are all continuous. For further details concerning direct and inverse systems and their limits see [24] and [42].

Proposition 9: Let $\{A_\alpha, h_\alpha^\beta\}$ be a direct system of algebras with direct limit A, and let $\{X_\alpha, \varphi_\alpha^\beta\}$ be the inverse system obtained by taking X_α to be the spectrum of A_α and φ_α^β to be the adjoint of h_α^β. Let X be the inverse limit of this system. Then X is the spectrum of A and the projections φ_α of X into X_α are the adjoints of the projections h_α of A_α into A.

Proof: Let $x \in X$, and $f \in A$. Then there is an α and an $f_\alpha \in A_\alpha$ such that $f = h_\alpha(f_\alpha)$. Define $x(f)$ to be $\varphi_\alpha(x)(f_\alpha)$. Then $x(f)$ is readily seen to be independent of the choice of α and to define a homomorphism of A onto the complex numbers, i.e. $x \in \Sigma(A)$. On the other hand, each $y \in \Sigma(A)$ comes from that $x \in X$ whose coordinates $\{x_\alpha\}$ are given by $x_\alpha \pi \circ h_\alpha$. Thus we have a natural correspondence between X and $\Sigma(A)$, and it follows from this construction that the projections φ_α and h_α are adjoint.

The principal use we make of direct systems is the following: Let A be an algebra and D the set of all finite subsets of A directed by inclusion $\subset$. If $\alpha = \{f_1, \ldots, f_n\}$ we let A_α be the subalgebra of A generated by α. For $\alpha \subset \beta$, we have $A_\alpha \subset A_\beta$, and we take h_α^β to be the inclusion map of A_α into A_β. Then $\{A_\alpha, h_\alpha^\beta\}$ is a direct system whose direct limit is A and for which the projection $h_\alpha : A_\alpha \to A$ is just the inclusion of A_α in A.

By the subalgebra A_α "generated" by α we may mean either (i) the subalgebra consisting of all polynomials $P(f_1, \ldots, f_n)$ of the elements of α, (ii) the subalgebra consisting of all elements g in A such that $g = P\,Q^{-1}$ where P and Q are polynomials in $f_1, \ldots, f_n$, and Q has an inverse in A, or (iii) the closure of the subalgebra in (i) in the case when A is a Banach algebra. For our purposes it is more convenient to take one of the latter notions, since we would like for A_α to have a compact spectrum if A does. In (iii) the algebras A_α are themselves Banach algebras.

4. Some examples of function algebras. In this section we give some examples of function algebras which illustrate the various properties discussed in the earlier sections.

Example 1: Let X be an open set in the plane or on a non-compact Riemann surface, and let A be the algebra of all analytic functions on X. Then A has properties (α), (β), (γ) and (δ). (cf. [29], [54], [55]). Thus it follows by Proposition 6 that X is the spectrum of A. If Y is another such open set on a Riemann surface and B the algebra of all analytic functions on Y, then by Proposition 8 every homomorphism h of A into B arises from a unique continuous mapping φ of Y into X so that $(hf)(y) = f(\varphi(y))$. If we make use of its continuity, the mapping φ is easily seen to be analytic and we have the results of [54] and [55].

Example 2: Let X be a Stein manifold [19] and A the algebra of analytic functions on A. Then the algebra A satisfies (β) (cf. [19], p. 11), and it can be shown that the spectrum of A is X ([22], p. 512). It follows from Corollary 4 of [19] that A satisfies property (δ). If Y is another Stein manifold and B the algebra of analytic functions on B, then every homomorphism h of A into B is, by Proposition 8, induced by a continuous mapping φ of Y into X such that $(hf)(y) = f(\varphi(y))$. We thus obtain the principal theorem of [22].

The discussion in Examples 1 and 2 shows the utilization of Proposition 8 to get the continuity of the mapping φ without knowing that the Gelfand topology for X coincides with the original topology

of X. Remmert has recently shown that every non-compact Riemann surface can be properly (in the sense of Bourbaki) embedded in C^3 and each Stein manifold in some C^n. From this it follows that the Gelfand topology for X in Examples 1 and 2 is in fact the original topology for X.

Example 3: An algebra in which (α_o) does not hold is the following: Let $X = [0,1]$ and let A consist of all continuous functions on $[0,1]$ which can be expressed in the form $P(x,e^x)/Q(x,e^x)$, where $P(x,y)$ and $Q(x,y)$ are polynomials in x and y with Q not divisible by y. Then the spectrum of A is X, while the ideal generated by e^x is a maximal ideal whose quotient field is isomorphic to the field of all rational functions of one indeterminate over the complex numbers. This ideal together with the points of $[0,1]$ constitute the set of maximal ideals of A.

Example 4: Let X be the bicylinder $\{< z_1, z_2 > : |z_1| \leq 1, |z_2| \leq 1\}$ in the space C^2 of two complex variables, and let A be the algebra consisting of those functions which are continuous on X and analytic in the interior of X. Then A satisfies (α), (β), and (γ), and its spectrum is X. Since X is compact, its topology is the Gelfand topology induced by A.

Example 5: Let Y be the boundary $\{< z_1, z_2 > : (|z_1| \leq 1, |z_2| = 1)$ or $(|z_1| = 1, |z_2| \leq 1)\}$ of the bicylinder, and let B be the algebra consisting of the restrictions to Y of the functions in the algebra A of the preceding example. Then B has property (α) but not (β). The spectrum $\Sigma(B)$ can be identified in a natural way with X, and $\hat{B}$ with A.

Example 6: Let X be the unit disc $|z| \leq 1$, and let A be the algebra of all functions which are analytic in $|z| < 1$ and continuous on X. Then the spectrum of A is X. Let A' be the algebra on X which consists of all uniform limits of polynomials in z and $|z|$. Then A' is a super-algebra of A, but the spectrum of A' is homeomorphic to the cone $|z| \leq t \leq 1$. If we let A'' be the algebra of all continuous functions on X, then A'' is a super-algebra of A', but its spectrum has shrunk back down to X again. This example is suggested by a remark in [51].

Example 7: Let X be the set of non-negative integers and A the algebra of all complex-valued sequences $< a_n >$ which are ultimately constant, i.e., those sequences for which there is an N such that $a_n = a_N$ for all $n > N$. Then A satisfies (β) but not (γ). The spectrum of A consists of X together with the homomorphism ω defined by $\omega(< a_n >) = \lim_{n \to \infty} a_n$.

Example 8: Let X be a completely regular space, and let A be the algebra of all continuous complex valued functions on X. Then A always satisfies (β), while the spectrum of A is the Q-closure of X introduced by Hewitt [34].

Example 9: Let A be the algebra of all bounded analytic functions on the unit disc $\{z : |z| < 1\}$ in the complex plane. This algebra and its spectrum have been discussed in some detail by I. J. Schark [63]. Further details are given in Hoffman [37].

Example 10: Let $U = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$, where the a_i's are real-valued differentiable functions of the x_i's. The set of all

differentiable functions defined in a region R and satisfying $Uf = 0$ there forms an algebra, and if R is sufficiently small, the spectrum of this algebra is just the set of characteristics of U lying in R. This set can be given a differentiable structure and then the algebra consists of all differentiable functions on the set of characteristics.

Example 11: If we let the coefficients of U in the preceding example be complex valued, then by [49] the algebra of solutions will in general have a spectrum which is a subset of the space of $(n-1)$ complex variables with the algebra represented as all functions continuous on the spectrum and complex analytic in the interior. If, for example, $U = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + i(x_1 + ix_2) \frac{\partial}{\partial x_3}$, then the spectrum of the algebra of functions which satisfy $Uf = 0$ in the sphere $x_1^2 + x_2^2 + x_3^2 \leq 1$ is the region in the space of two complex variables given by

$$\operatorname{Im} z_2 \leq - |z_1|^2$$
$$(\operatorname{Re} z_2)^2 + |z_1|^2 \leq 1 ,$$

and the algebra is represented by all functions which are complex analytic in this region and sufficiently smooth on the boundary. All solutions of the equation $Uf = 0$, are obtained by taking such a function and replacing z_1 by $x_1 + ix_2$ and z_2 by $x_3 - i(x_1^2 + x_2^2)$.

Further examples of function algebras and their properties are given by Hoffman and Singer [39]. Interesting special examples are treated by Bishop [10] and by Stolzenberg [67].

5. Boundaries for function algebras. Let A be a function algebra with compact spectrum $\Sigma(A)$. Šilov [26] has shown that there is a closed set $Y \subset \Sigma(A)$ with the property that $\sup_{y \in Y} |f(y)| = \sup_{x \in \Sigma(A)} |f(x)|$ and that if any other closed set $Z \subset \Sigma(A)$ has this property then $Y \subset Z$. This set Y is referred to as the Šilov boundary for A. If A is a function algebra on a compact set X and if A is weakly inverse closed (satisfies property (α_o)), then it follows from Proposition 1 that the Šilov boundary Y for A is contained in X. Numerous examples are given in [39].

Bishop [12] has shown that for a uniformly closed function algebra A on a compact metric space X the set M consisting of those points in X, each of which is the unique maximum of some f in A, has the property that $\sup_{x \in X} |f(x)| = \sup_{y \in M} |f(y)|$ for each f in A. Further properties of this Bishop boundary are discussed in [17].

6. Dirichlet algebras. A function algebra A on a compact set X is called a Dirichlet algebra [28] if the real parts of functions in the algebra are dense (in the sense of uniform convergence) in the space $C(X)$ of all continuous functions on X. Interesting results concerning Dirichlet algebras have been obtained by Wermer [75], [76] and a nice treatment of their properties can be found in Hoffman [37].

A generalization due to Hoffman [38] is the concept of a log-modular algebra which is an algebra A on a compact space X such that the logarithms of the absolute values of the invertible elements in A are dense in $C(X)$. Hoffman shows that many of the most important properties of Dirichlet algebras are also shared by the log-modular algebras.

7. The one-dimensional cohomology of the spectrum. This section is devoted to a proof of a theorem of Arens and myself which says that for certain algebras A the one-dimensional Cech cohomology group over the integers of the spectrum of A is isomorphic to the quotient of the group of units in A by the subgroup of exponentials. Proposition 12 states this result for a class of function algebras and the following theorem states that this is also true for every commutative Banach algebra. These theorems and their proofs are given here not only because of their intrinsic interest, but also because the proof gives an excellent illustration of the application of the theory of functions of several complex variables to obtain results in function algebras.

We begin by describing a canonical homomorphism η of the non-zero continuous complex-valued functions on a paracompact space X in the one-dimensional cohomology group $H^1(X,I)$ of X with integer coefficients. Let $\mathcal{C}$ be the sheaf of germs of continuous functions on X (with additive group structure), $\mathcal{C}^*$ the sheaf of germs of continuous non-vanishing functions on X (with multiplication as the group operation), and I the constant sheaf of integers on X. If exp is the mapping $f \to e^{2\pi i f}$, then we have the following exact sequence of sheaves

$$0 \longrightarrow I \longrightarrow \mathcal{C} \xrightarrow{\exp} \mathcal{C}^* \longrightarrow 0 .$$

Hence we have the exact sequence of cohomology groups [35]:

$$0 \longrightarrow H^0(X,I) \longrightarrow H^0(X,\mathcal{C}) \longrightarrow H^0(X,\mathcal{C}^*) \xrightarrow{\eta} H^1(X,I)$$
$$\longrightarrow H^1(X,\mathcal{C}) = 0 ,$$

the last group being 0 since $\mathcal{C}$ is a fine sheaf [35]. Thus we have a homomorphism η of the multiplicative group of continuous functions on X which do not vanish anywhere onto the group $H^1(X,I)$.(cf.[23]). The kernel of η consists of the range of the exponential mapping, i.e. of those g for which there is a continuous function f such that $g = \exp f$. Note that if g is a continuous function on X and $|g - 1| < \epsilon < 1$, then g has a continuous logarithm, and $\eta g = 0$.

It is readily verified that the mapping η is functorial in the sense that, if Y is another paracompact space, then for each continuous mapping $\varphi : Y \to X$ the adjoint mappings $\varphi^* : H^1(X,I) \to H^1(Y,I)$ and $\varphi^* : C(X) \to C(Y)$ commute with η, i.e. $\eta\varphi^* = \varphi^*\eta$.

We shall need two results from the theory of several complex variables about analytic functions on polynomially convex sets in C^n. A compact set X in the space C^n of n-complex variables is said to be polynomially convex if given $y \notin X$, there is a polynomial p such that

$$|p(y)| > \sup_{x \in X} |p(x)| .$$

If $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X, Theorem B of Cartan [19] states that $H^1(X,\mathcal{O}) = 0$. From this we derive the following lemma:

Lemma 2: Let X be a compact polynomially convex set in C^n. Then the multiplicative group of non-vanishing holomorphic functions (i.e. of continuous functions which have an extension which is holomorphic in a neighborhood of X) is mapped onto $H^1(X,I)$ by the

mapping η. The kernel of η consists of functions which are exponentials of holomorphic functions on X.

Proof: Let $\mathcal{O}^*$ be the sheaf of germs of non-vanishing holomorphic functions on X. Then we have the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0 ,$$

from which we obtain the exact sequence

$$\longrightarrow H^0(X, \mathcal{C}) \xrightarrow{\exp} H^0(X, \mathcal{C}^*) \xrightarrow{\eta} H^1(X,I) \longrightarrow H^1(X,\mathcal{C}) = 0 ,$$

the last group being zero by Theorem B. But $H^0(X, \mathcal{C}^*)$ is just the multiplicative group of non-zero holomorphic functions on X, and the exactness of the sequence is precisely the conclusion of the lemma.

We shall also need the following lemma. This lemma is standard and is an easy consequence of corollaire 4 of [19], and we omit the proof.

Lemma 3: Let X be a compact polynomially convex set in C^n and f a holomorphic function on X, i.e. on a neighborhood of X. Then f can be uniformly approximated on X by polynomials.

If A is a function algebra, the units in A are the functions which are never zero on the spectrum X of A. Thus η maps the units in A into $H^1(X,I)$. We are now in a position to prove a preliminary proposition:

Proposition 10: Let A be a finitely generated algebra with compact spectrum X, and suppose that A is closed under uniform

convergence on X. Then η maps the group M of units in A onto $H^1(X,I)$ and the kernel of η is the range E of the exponential function on A.

Proof: Let $f_1, \ldots, f_n$ be the generators of A. Then A consists of those functions on X which are uniform limits of polynomials in $(f_1, \ldots, f_n)$. Let F be the mapping of X into C^n whose components are $f_1, \ldots, f_n$. Since A separates points of X, and each element in A is a uniform limit of polynomials in tne f_i, it follows that the f_i separate points in X. Thus F is a homeomorphism of X onto a compact set in C^n, and if we identify X with $F[X]$, the algebra A becomes the algebra of uniform limits of polynomials on a compact subset X of C^n. If for some $y \in C^n$ we have

$$|p(y)| \leq \sup_{x \in X} |p(x)| ,$$

then the homomorphism π defined on polynomials by

$$\pi(p) = p(y)$$

extends to a homomorphism on A. Thus y is in the spectrum of A. Since X is the spectrum of A there is an x in X such that $p(y) = p(x)$ for all polynomials p. But this implies $y = x$ and hence $y \in X$. Thus X is a polynomially convex subset of C^n.

It follows from Lemma 3 that all functions which are holomorphic on X are in A. Since the group M of units of A is the set of functions in A which are never zero, M contains all holomorphic

non-vanishing functions on X, and Lemma 2 implies that η maps A onto $H^1(X,I)$.

Let g be a function A which is never zero on X, and let $\epsilon = \inf |g(x)|$ on X. Since A is the closure on X of the polynomials, there is a polynomial p on X such that $|g - p| < \epsilon/2$. Thus p is a non-vanishing holomorphic function on X, and $p = g[1 - (g-p)/g]$. Since $|(g-p)/g| < 1/2$, we have $\eta[1 - (g-p)/g] = 0$, and consequently $\eta(p) = \eta(g) = 0$. Hence by Lemma 2 there is a holomorphic function h on X such that $p = \exp h$. Since A is uniformly closed, the series

$$\sum \frac{(g-p)^n}{ng^n}$$

converges to a function k in A, and $1 - (g-p)/g = \exp(-k)$. Thus $g = \exp(h + k)$. This shows that the kernel of η is the subgroup of exponentials, proving the proposition.

The next proposition removes the restriction that A be finitely generated. The method of proof is typical of the manner in which propositions about finitely generated algebras can be generalized.

Proposition 11. Let A be a uniformly closed function algebra with compact spectrum X. Then η maps the group M of units of A onto $H^1(X,I)$ and the kernel of η consists of range E of the exponential mapping on A.

Proof: Let $\{A_\alpha\}$ be the system of finitely generated[1] sub-algebras of A directed under inclusion. If $\alpha \prec \beta$, i.e. if $A_\alpha \subset A_\beta$, we define h_α^β to be the inclusion mapping. Then A is the direct limit of the direct system $\{A_\alpha; h_\alpha^\beta\}$ of algebras.

Let X_α be the spectrum of A_α, and φ_α^β be the adjoint of h_α^β. Then by Proposition 9 the spectrum X of A is the inverse limit of $\{X_\alpha, \varphi_\alpha^\beta\}$, and the projections φ_α of X onto X_α are the adjoints of the projections h_α of A_α into A. Moreover [42], the group $H = H^1(X,I)$ is the direct limit of the direct system of groups $\{H_\alpha, \varphi_\alpha^{*\beta}\}$ where $H_\alpha = H^1(X_\alpha, I)$. Thus if $\gamma \in H$, there is an α and an element $\gamma_\alpha \in H_\alpha$ such that $\gamma = \varphi_\alpha^* \gamma_\alpha$. By Proposition 10 there is an $f \in A_\alpha$ such that $\gamma_\alpha = \eta f$. Thus

$$\gamma = \varphi_\alpha^* \eta f = \eta \varphi_\alpha^* f = \eta(f \circ \varphi_\alpha)$$

But $f \circ \varphi_\alpha$ is a non-vanishing function in A. Thus η maps M onto $H^1(X,I)$.

Suppose now that f is a unit in A and that $\eta f = 0$. Then f is also a unit in the algebra A_α generated by f and f^{-1}. Let f_α be the element f of A_α considered as a function on the spectrum X_α of A_α. Then $f = h_\alpha f_\alpha$, and

$$0 = \eta f = \varphi_\alpha^* \eta f_\alpha .$$

---

[1] Here we mean an algebra A_α which is the uniform closure of all polynomials in a given finite set of elements of A.

Since H is the direct limit of the groups H_β, there must be a $\beta \succ \alpha$ such that

$$0 = \varphi_\alpha^{*\beta} \, \eta f_\alpha \, .$$

Hence

$$0 = \eta(h_\alpha^\beta \, f_\alpha) \, .$$

Thus $h_\alpha^\beta f_\alpha$ is an element of A_β which is mapped into zero by η. Thus by Proposition 10 there is a g in A_β such that $h_\alpha^\beta f_\alpha = \exp g$. But this implies that $f = h_\beta \, h_\alpha^\beta f = h_\beta \exp g = \exp(g \circ \varphi_\beta)$. Thus f is the exponential of an element of A, proving that the kernel of η is the range of the exponential function on A.

We now extend this last proposition to a class of algebras which we do not assume to be uniformly closed.

Proposition 12: Let A be an algebra with compact spectrum X, and suppose that A as a function algebra on its spectrum is inverse closed. Then η maps the group M of units of A onto $H^1(X,I)$. Suppose further that for each $f \in A$ the function $\exp f$ is in A. Then the kernel of η is the uniform closure of the range of the function $\exp$ on A. If in addition A has the property that if $|f - 1| < 1/2$ then $f = \exp g$ for $g \in A$, then the kernel of η is just the range E of the exponential function on A. Thus if A satisfies these two conditions we have

$$H^1(X,I) \cong M/E \, .$$

Proof: Let $\bar{A}$ be the algebra on X which is the uniform closure of A. Since A is inverse closed, X is also the spectrum of $\bar{A}$ by Proposition 3. For each $\gamma \in H^1(X,I)$ there is a non-vanishing function f in $\bar{A}$ such that $\eta f = \gamma$. Let $\epsilon = \min |f|$, and let g be an element of A such that $|f - g| < \epsilon/2$. Then g does not vanish on X, and

$$\begin{aligned}\eta g &= \eta f + \eta[1 - (f-g)/f] \\ &= \eta f ,\end{aligned}$$

since $\eta[1 - (f-g)/f]$ must be zero because $|(f-g)/f| < 1/2$. This shows that η is onto $H^1(X,I)$.

Suppose that f is a non-vanishing function in A and $\eta f = 0$. Then by Proposition 11 there is a function $h \in \bar{A}$ such that $f = \exp h$. Since h can be approximated by elements in A, f can be approximated by exponentials of functions in A. Thus if the exponential function is defined on A, we have the kernel of η equal to the uniform closure of the range of exp.

In particular, if $\eta f = 0$, we can find a g in A such that $|f - \exp g| < 1/3 \min f$. Then $|1 - f \exp(-g)| < 1/2$. But if A satisfies the last condition of the proposition, then
$f \exp(-g) = 1 - [1 - f \exp(-g)] = \exp(h)$ for some h in A. Then $f = \exp(g + h)$, and f is in the range of exp. This proves the proposition.

Even if A is not a function algebra, we can define a mapping η of the group of units of A into $H^1[\Sigma(A),I]$ by defining ηf to be $\eta \hat{f}$, where $\hat{f}$ is the representation of f as a function on $\Sigma(A)$.

The following theorem (due to Arens and myself) shows that the conclusion of the last proposition is true for arbitrary commutative Banach algebras.

Theorem: Let A be a commutative Banach algebra with spectrum X. Let M be the group of units in A and E the range of the exponential function on A. Then

$$H^1(X,I) \cong M/E \ .$$

Proof: Let $f \to \hat{f}$ be the mapping which associates to each f in A the function $\hat{f}$ on X defined by $\hat{f}(\pi) = \pi f$. Let $\hat{A}$ be the image of A under this mapping. Since X is the space of maximal ideals of A, the algebra $\hat{A}$ is inverse closed and each nowhere zero function in $\hat{A}$ is the image of a unit in A. Thus η maps the group M of units of A onto $H^1(X,I)$ by Proposition 12. Since A is a Banach algebra, the function exp is defined and takes elements of $\hat{A}$ into elements of $\hat{A}$. Thus by Proposition 12 a unit f of A is mapped into zero by η if and only if $\hat{f}$ is the uniform limit of functions $\hat{g}$ where $g = \exp h$ with h in A. Choose g of this form such that $\sup |\hat{f} - \hat{g}| < 1/3 \inf |\hat{f}|$. Then $|1 - \hat{f}\hat{g}^{-1}| < 1/2$, and so [50] we have

$$\overline{\lim} \, \|(1 - fg^{-1})^n\|^{1/n} \leq 1/2 \ .$$

Thus the series

$$\sum_{n=1}^{\infty} \frac{(1 - fg^{-1})^n}{n}$$

converges in A to an element k, and we have $fg^{-1} = \exp(-k)$. Thus $f = \exp(h - k)$, and f is in the range of the exponential function.

The method used to derive Proposition 10 from Cartan's Theorem A can also be used to derive the following proposition about certain algebras whose spectra are not compact. Remembering that every open Riemann surface (and a fortiori every plane domain) is a Stein manifold, this proposition gives a result of Rudin's as a special case [58].

Proposition 13: Let A be the algebra of all holomorphic functions on a Stein manifold X, let M be the multiplicative group of nowhere vanishing functions in A and let E be the range of the exponential function on A. Then

$$H^1(X, I) \cong M/E .$$

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